Homework 6, due 4/3

Only your **four** best solutions will count towards your grade.

1. Let $* : \mathcal{A}^{p,q} \to \mathcal{A}^{n-p,n-q}$ denote the Hodge star operator defined in class.
   (a) Show that $* * \alpha = (-1)^{p+q}\alpha$ for $\alpha \in \mathcal{A}^{p,q}$.
   (b) Show that $*$ is a conjugate linear isometry for the Hermitian inner product, i.e. $\langle *\alpha, *\beta\rangle = \langle \beta, \alpha\rangle$ for all $\alpha, \beta \in \mathcal{A}^{p,q}$.

2. Let $T = \mathbf{C}/(\mathbf{Z} \oplus \sqrt{-1}\mathbf{Z})$ be a torus. Show directly from the definitions that the deRham cohomology of T is given by $H^0(T, \mathbf{R}) \cong \mathbf{R}$, $H^1(T, \mathbf{R}) \cong \mathbf{R}^2$ and $H^2(T, \mathbf{R}) \cong \mathbf{R}$.

3. Let (X, g) be a connected Kähler manifold of dimension at least 2. Show that if for a function $f : X \to \mathbf{R}$ the metric $e^f \cdot g$ is Kähler, then necessarily f is constant.

4. Let ω denote the standard Kähler form on $\mathbf{C}$, i.e. $\omega = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$. Show that for a function f on $\mathbf{C}$ we have
$$(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)f = -\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

5. Let $f : D \to \mathbf{C}$ denote a locally integrable function on the unit disk in $\mathbf{C}$, which is holomorphic in a distributional sense, i.e. $(f, \bar\partial^*\alpha)_{L^2} = 0$ for all smooth $(0,1)$-forms α compactly supported in D. We are using the standard metric on $\mathbf{C}$ to define the adjoint, and $(f, g)_{L^2}$ denotes the L^2-inner product
$$(f, g)_{L^2} = \int_D f\bar{g}\,\mathrm{Vol},$$
using the standard volume form on $\mathbf{C}$. In this and the following problem we will show that f is smooth and holomorphic. First show that if we already know that f is smooth, then f is necessarily holomorphic.

6. Consider the setup of the previous question. Let $\rho : \mathbf{C} \to \mathbf{R}$ denote a smooth radially symmetric function, supported in D, such that $\int_D \rho\,\mathrm{Vol} = 1$. For $\epsilon > 0$ define ρ_ϵ by $\rho_\epsilon(z) = \epsilon^{-2}\rho(\epsilon^{-1}z)$, and define f_ϵ to be the convolution
$$f_\epsilon(z) = \int_{\mathbf{C}} f(z-w)\rho_\epsilon(w)\,\frac{1}{2}\sqrt{-1}dw \wedge d\bar{w},$$
for $|z| < 1 - \epsilon$.
   (a) Show that f_ϵ is smooth and holomorphic.
   (b) By bounding the integral of f_ϵ, show that once ϵ is small enough ($\epsilon < 1/4$ is enough), then we have a uniform derivative bound $|f'_\epsilon| < C$ on $D_{1/2}$, independent of ϵ.
   (c) Deduce that along a sequence $\epsilon_k \to 0$ we can extract a holomorphic limit $f_{\epsilon_k} \to f_0$ on $D_{1/2}$, and necessarily $f_0 = f$.